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The classical problem of the stability of a pair of vortices behind a circular cylinder in inviscid incompressible flow originally studied by Föppl is revisited. A correction over Föppl's analysis is made regarding the stability of the vortices under small symmetric perturbations. A simpler and more general stability condition for any vortices in a two-dimensional flow field is presented. This new condition can be easily tested analytically or by numerical computation when analytical expressions are difficult to obtain. The new condition is then applied to study the stability of vortices behind elliptic cylinders and circular cylinders with a splitter plate at the rear stagnation point. Stationary positions of the symmetric vortices are made unique by postulating the separation point on the body. The effect of the size of the splitter plate on the stability of the vortices is studied. Results agree well with known experimental observations.

I. Introduction

The problem of the stability of a vortex pair behind a circular cylinder is a classical problem originally studied by Föppl (1913)[1] and cited by many authors including Lamb (1932),[2] Milne-Thomson (1968),[3] Goldstein (1938),[4] and Saffman (1992, p.43).[5] Aside from its theoretical significance, this problem is also of practical use in studying vortex trapping above slender wings or bodies. Free vortices trapped over a delta wing, a fuselage, or a wing-body combination at high angles of attack are known to greatly increase the lift coefficient. However, the stability of such vortices are of major concern in using the idea of vortex trapping. By using the slender body theory, one may reduce the three-dimensional vortex problem over a slender body into a modified two-dimensional problem of a pair of vortices trapped behind a two-dimensional cross-section with the addition of terms due to three-dimensionality. Therefore, Föppl's classical results on the stability of a pair of vortices behind a circular cylinder provides a starting point for analysis of more complex two- and three-dimensional problems.

In this paper we will revisit the problem of the circular cylinder and point out a correction on the result by Föppl (1913).[1] A general stability condition for vortex motion in an arbitrary two-dimensional flow under small perturbations is then presented. This condition involves only the evaluation of the divergence and Jacobian of a vortex velocity field and, therefore, can be easily tested either analytically or by numerical computation when analytical expressions are difficult or impossible to obtain. A computational method using a finite-difference method to evaluate the divergence and Jacobian in testing the new stability condition is validated for the circular cylinder case where analytic solutions are available. The inviscid model for the stability analyses of the vortex pair separated from a bluff body is briefly reviewed and the condition at the separation point is presented. The new stability condition is then used to study the stability of a pair of vortices behind a circular cylinder with a splitter plate. The stability condition is tested numerically for this case to avoid analytical complexities. It is found that there is a minimum plate length that makes the vortices neutrally stable under anti-symmetric perturbations. The stationary positions and stability of a pair of vortices behind an elliptical cylinder is also investigated.

The extensions of the present investigation to study the stability of a vortex pair over three-dimensional slender conical bodies are given in a companion paper (Cai et al. 2001).[6]

II. Vortices behind a Circular Cylinder

Föppl (1913)[1] studied the stability of a pair of vortices behind a circular cylinder in a uniform inviscid incompressible stream of velocity U as shown in Fig. 1. Following Föppl's notation, the pair of vortices are assumed to be at $\zeta_0 = \xi_0 + i\eta_0$ and $\overline{\zeta}_0 = \xi_0 - i\eta_0$ on the complex domain. The complex potential of the flow

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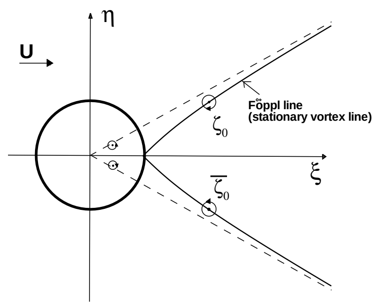

Fig. 1 Vortices behind a Circular Cylinder and Föppl Line.

field can be found by placing two image vortices inside the cylinder and using the principle of superposition as follows.

$$W = \Phi + i\Psi = U(\zeta + \frac{1}{\zeta}) + iCln\frac{(\zeta-\zeta_0)(\zeta-1/\zeta_0)}{(\zeta-\bar{\zeta}_0)(\zeta-1/\bar{\zeta}_0)} \quad (1)$$

where $C = \frac{\Gamma}{2\pi}$ and Γ is the strength (circulation) of the vortex and the cylinder radius is assumed to be one. The complex velocity is then

$$\begin{aligned} u - iv &= U\left(1-\frac{1}{\zeta^2}\right) + iC\left[\frac{1}{\zeta-\zeta_0} + \frac{1}{\zeta-1/\zeta_0}\right. \\ &\quad \left. -\frac{1}{\zeta-\bar{\zeta}_0} - \frac{1}{\zeta-1/\bar{\zeta}_0}\right] \end{aligned} \quad (2)$$

The velocity of the vortex at ζ_0 can be found by removing the induced velocity term due to the vortex at ζ_0 itself. This is justified by replacing the point vortex by the finite cross section of the Rankine vortex filament (see for example Saffman 1992,[5] p.22). At the center of the vortex filament cross section, the induced velocity by the vortex itself is zero. For clarity, the subscript 0 is dropped hereupon. Thus, the flow velocity at the center of the vortex whose location is at $\zeta = \xi + i\eta$ is

$$\begin{aligned} u - iv &= U\left(1-\frac{1}{\zeta^2}\right) + iC\left[\frac{1}{\zeta-1/\zeta}\right. \\ &\quad \left. -\frac{1}{\zeta-\bar{\zeta}} - \frac{1}{\zeta-1/\bar{\zeta}}\right] \end{aligned} \quad (3)$$

This may be written in terms of ξ, η, and $r = \sqrt{\xi^2+\eta^2}$ as

$$\begin{aligned} u &= U\left(1-\frac{\xi^2-\eta^2}{r^4}\right) + C\eta\left[-\frac{1}{2\eta^2}\right. \\ &\quad \left. + \frac{2(r^2-2\eta^2-1)}{(r^4-2r^2+4\eta^2+1)(r^2-1)}\right] \end{aligned} \quad (4)$$

$$-v = \frac{2U\xi\eta}{r^4} - \frac{4C\xi\eta^2}{(r^4-2r^2+4\eta^2+1)(r^2-1)} \quad (5)$$

For the vortices to be stationary ζ cannot be allowed to have arbitrary values since the flow velocity at ζ must be zero. By setting u and v to zero, it is found that any pair of stationary vortices must be located on the following curves,

$$\eta_0 = \pm\frac{(r_0^2-1)}{2r_0} \quad (6)$$

These curves are called Föppl lines(see solid lines in Fig. 1). Points on the Föppl lines are called stationary points. Vortices located at these points will not move. The strength of the vortices C varies with their location along these curves,

$$C = 2\eta_0\left(1-\frac{1}{r_0^4}\right)U = \frac{(r_0^2-1)^2(r_0^2+1)}{r_0^5}U \quad (7)$$

The question of stability arises when vortices are slightly perturbed from these stationary points. Föppl studied this problem by decomposing any arbitrary such perturbation into a symmetric and an anti-symmetric perturbation. He found that the vortices are stable for symmetric perturbations, but unstable for anti-symmetric perturbations. Unfortunately, his analysis on the symmetric perturbations was flawed. On re-examination of the problem, it is found that the vortices are only neutrally stable for symmetric perturbations. To illustrate this, the analysis on the symmetric perturbations is discussed below in detail.

Notice that Eqns. (3), (4), and (5) hold true as long as the vortex pair is located symmetrically with respect to the ξ axis. Therefore, it is true not only when the vortices are stationary, but also when they are perturbed symmetrically. The strength of the vortices may be assumed to remain unchanged when their positions are slightly perturbed. Consider such a symmetric perturbation where, say, the top vortex is perturbed from its original stationary location $\zeta_0 = (\xi_0, \eta_0)$ to a new location $(\xi_0+\alpha, \eta_0+\beta)$, where $|\alpha|, |\beta| \ll 1$. The velocity of the motion of the vortex center is obtained by directly substituting $\xi_0+\alpha$ for ξ and $\eta_0+\beta$ for η in Eqns. (4) and (5), respectively, and then expanding and keeping only the linear terms of α and β. Use is also made of the fact that ξ_0 and η_0 must satisfy Eqns. (6) and (7). Thus, we have

$$\frac{d\alpha}{dt} = u = A\alpha + B\beta \quad (8)$$

$$-\frac{d\beta}{dt} = -v = X\alpha + Y\beta \quad (9)$$

where t denotes time, and

$$A = Y = -\frac{U\xi_0}{r_0^8}(r_0^4 - 3r_0^2 + 2) \quad (10)$$

$$B = \frac{U\eta_0}{r_0^8(r_0^2-1)}(4r_0^8 + 5r_0^6 + 2r_0^4 - 5r_0^2 + 2) \quad (11)$$

$$X = \frac{4U\xi_0^2\eta_0}{r_0^6(r_0^4-1)}(r_0^4 + r_0^2 + 2) \quad (12)$$

The general solution of the above system of equations can be written as

$$\alpha = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{13}$$
$$\beta = C_3 e^{\lambda_1 t} + C_4 e^{\lambda_2 t} \tag{14}$$

where λ_1 and λ_2 are the two eigenvalues of the system, and C_1, C_2, C_3, and C_4 are arbitrary constants independent of t. The vortices will be stable if both λ_1 and λ_2 have negative real parts, neutrally stable if both λ_1 and λ_2 are imaginary, and unstable if at least one of the eigenvalues has a positive real part.

For the above symmetric perturbations, we have

$$\lambda_{1,2} = \pm i\sqrt{BX - A^2} \tag{15}$$

where

$$BX - A^2 = \frac{4U^2\xi_0^2}{r_0^8(r_0^2+1)}(r_0^4 + 2r_0^2 + 5) \tag{16}$$

which is always positive. Therefore, a vortex pair behind the circular cylinder is neutrally stable under symmetric perturbations around its stationary point. It is noted that the neutral stability indicates a situation where the vortices oscillate around their stationary locations with an angular velocity between 0 when $r_0 \to \infty$ and $4U/a$ (where a is the radius of the cylinder) when $r_0 = 1$.

Föppl, however, found in 1913[1] that the eigenvalues have negative real parts and thus concluded that the vortices are absolutely stable under symmetric perturbations. Notice that in arriving at Eqns. (15) and (16), we substitute $\xi = \xi_0 + \alpha$ and $\eta = \eta_0 + \beta$ into Eqns. (4) and (5) first before we apply Eqns. (6) and (7) for the stationary points ξ_0 and η_0. Föppl, however, erroneously applied the stationary point conditions, Eqns. (6) and (7), to the arbitrary vortex location ξ and η and substituted them in the vortex velocity Eqns. (4) and (5) before applying the perturbations around the stationary points, ie., $\xi = \xi_0 + \alpha$ and $\eta = \eta_0 + \beta$. This led to different coefficients A, B, X, and Y in Equations (8) and (9), and subsequently different eigenvalues λ_1 and λ_2, and thus the erroneous conclusion that the vortices are absolutely stable under symmetric perturbations.

The same mistake is not found in his analysis for anti-symmetric perturbations. Thus, his result that the vortices are unstable for anti-symmetric perturbations still stands. It is interesting to note that Lamb, in his classical book (1932),[2] cited only Föppl's result on anti-symmetric perturbations when discussing Föppl's paper (1913).[1] So did Saffman in his book (1992, p.43).[5] Saffman also noted in a footnote on the same page that "I. Soibelman has pointed out (private communication) that the algebraic formulae given by Föppl for the eigen frequencies are in error, but the qualitative considerations are correct." In the course of deriving the formulas presented in this paper and also validating the results through computations, it is found that Eqn. (17) in Föppl's paper (1913)[1] does contain a small misprint. The last term in his expression for X' should not contain the factor η. The correct expression is

$$X' = \frac{2U\eta}{r^6}\left\{2(\eta^2 - \xi^2) - \frac{r^2-1}{2} - 2r^2(r^2+1)\right\} \tag{17}$$

The conclusions regarding stability for the anti-symmetric perturbations, indeed, do not change.

III. A General Stability Condition

The analysis in the last section can be generalized to yield a generalized stability condition for the motion of a vortex or a group of vortices. Consider a system of vortices in a two-dimensional flow. Assume one of the vortices in the system is located at (x, y). As this vortex is moved in the physical plane, other vortices in the system are assumed to move according to a given mode of motion, for instance, the symmetric or anti-symmetric mode of motion in the above section, subject to given boundary conditions. The starting point for the general stability condition to be discussed below is that given the flow boundary conditions and the mode of vortex motion, one has already obtained the velocity (u, v) for the vortex under consideration as a function of its location (x, y) similar to Eqns. (4) and (5) in the last section for the case of the symmetric vortex pair behind a circular cylinder. Thus, we write in general

$$\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases} \tag{18}$$

The stationary points (x_0, y_0) for the vortex can be found by solving

$$\begin{cases} u(x_0, y_0) = 0 \\ v(x_0, y_0) = 0 \end{cases} \tag{19}$$

Since the velocity function $u(x, y)$ and $v(x, y)$ must be analytic at (x_0, y_0), we have, for example,

$$u(x_0 + \Delta x, y_0 + \Delta y) = u(x_0, y_0) + \left(\frac{\partial u}{\partial x}\right)_0 \Delta x + \left(\frac{\partial u}{\partial y}\right)_0 \Delta y + O(\Delta x^2, \Delta y^2) \tag{20}$$

where $(\ )_0$ denotes the value of $(\ )$ at (x_0, y_0). By using the stationary point condition (19) and neglecting higher order terms, we get

$$u(x_0 + \Delta x, y_0 + \Delta y) = \left(\frac{\partial u}{\partial x}\right)_0 \Delta x + \left(\frac{\partial u}{\partial y}\right)_0 \Delta y \tag{21}$$

	D_0	J_0	Comment
Stable	< 0	> 0	
Neutral	< 0	$= 0$	non-oscillating
	$= 0$	$= 0$	non-oscillating
	$= 0$	> 0	oscillating
Unstable	> 0	any	
	any	< 0	

Table 1 Stability Condition for Vortex Motion.

When the vortex is perturbed from its stationary point (x_0, y_0) and then let go, its motion is assumed to follow the flow velocity given in the above equation and its displacement represented by Δx and Δy. Thus, we have

$$\begin{aligned} \frac{d}{dt}(\Delta x) &= u(x_0 + \Delta x, y_0 + \Delta y) \\ &= \left(\frac{\partial u}{\partial x}\right)_0 \Delta x + \left(\frac{\partial u}{\partial y}\right)_0 \Delta y \end{aligned} \qquad (22)$$

Similarly,

$$\begin{aligned} \frac{d}{dt}(\Delta y) &= v(x_0 + \Delta x, y_0 + \Delta y) \\ &= \left(\frac{\partial v}{\partial x}\right)_0 \Delta x + \left(\frac{\partial v}{\partial y}\right)_0 \Delta y \end{aligned} \qquad (23)$$

The above two equations form a system of first-order linear differential equations for Δx and Δy as a function of time t, i.e.,

$$\frac{d}{dt}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} (\frac{\partial u}{\partial x})_0 & (\frac{\partial u}{\partial y})_0 \\ (\frac{\partial v}{\partial x})_0 & (\frac{\partial v}{\partial y})_0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \qquad (24)$$

The solution to this system of equations has the same form as shown in Eqns. (13) and (14), where λ_1 and λ_2 are now the eigenvalues of the coefficient matrix in the above equation. Define the Jacobian and divergence of the vortex velocity field $\mathbf{q} = (u, v)$,

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}, \quad D = \nabla \cdot \mathbf{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \qquad (25)$$

It can be easily shown that the eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(D_0 \pm \sqrt{D_0^2 - 4J_0}\right) \qquad (26)$$

where the subscript 0 denotes values at (x_0, y_0). The stability condition for the vortex motion may then be summarized in Table 1.

Once the vortex velocity field $u(x, y)$ and $v(x, y)$ are given, it is straight forward to calculate, either analytically or numerically, the divergence D and the Jacobian J at any given stationary points. The stationary points can be found by solving Eqns. (19). A numerical method may also be used for this purpose.

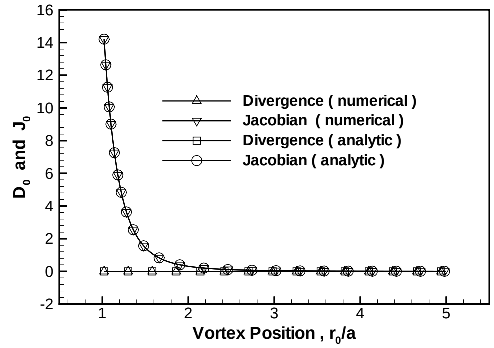

Fig. 2 Divergence and Jacobian vs. vortex position under symmetric perturbations, circular cylinder case.

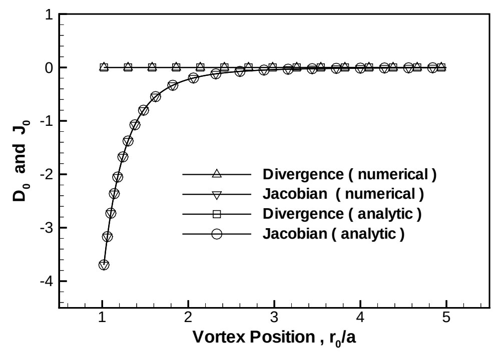

Fig. 3 Divergence and Jacobian vs. vortex position under anti-symmetric perturbations, circular cylinder case.

For the circular cylinder problem in the previous section, the analytical form of $u(x, y)$ and $v(x, y)$ is relatively simple so that both the stationary points and the stability conditions can be found analytically. By explicitly evaluating the D_0 and J_0 for this case, one can easily verify that Eqn. (26) gives the same result as Eqn. (15). For more complex problems, such as the flow over a circular cylinder with a splitter plate and that over ellipses to be presented later in this paper, numerical search for the stationary points and evaluation of the stability conditions may be more convenient or necessary where analytical expressions are difficult to obtain. A simple bisecting root-finding algorithm and the standard fourth order central finite-difference formula are used in this paper.

As an example and a means of validating our computational method, numerical computations are performed and compared with the analytical results for the circular cylinder case. Figs. 2 and 3 show numerical and analytical values of D_0 and J_0 versus vortex position r_0/a (a is the radius of the circular cylinder) for the symmetric and anti-symmetric perturbations, respectively. For simplicity, D_0 and J_0 here and from this point on denote $D_0 a/U$ and $J_0 a^2/U^2$, respectively. The computational results are indistinguishable from the analytical values. Clearly, the divergence D_0 for both the symmetric and anti-symmetric perturbations are zero. For the case of symmetric perturbations, it is found that $J_0 > 0$, which indicates that the vortices are in a neutrally stable state with an oscillatory motion near the stationary points. The oscillation attenuates as the vortices move farther from the cylinder. For the anti-symmetric perturbations, it is found that $J_0 < 0$, thus the vortices are always unstable. The vortices are more unstable when they are close to the cylinder and gradually approach neutral stability as they move away from the cylinder.

It is noticed that both the analytic and computational results show that the divergence of the vortex velocity field D_0 is zero for both the symmetric and the anti-symmetric perturbations. This can be easily proven analytically for the circular cylinder case. It is observed numerically that this property is true for a number of flow fields that are obtained from the circular cylinder flow through a series of conformal mappings. It appears that this is true for all two-dimensional incompressible flows pending an analytical proof. Hence, it may be conjectured that vortices in any two-dimensional incompressible flow is either unstable or neutrally stable, but never absolutely stable, based on the above table.

IV. Relevance of an Inviscid Model with Concentrated Vortices

The basic flow pattern of separated vortex flow behind a bluff body is known from many experimental observations. For example, Van Dyke documented pictures of flow around a circular cylinder for a Reynolds number of 2000 (Van Dyke 1982,[7] Fig. 47, p. 31). The left part of Fig. 4 illustrates the vortex structure behind a circular cylinder. A pair of vortex sheets erupt from the separation points S_1 and S_2. They extend downstream and then roll tightly into two concentrated vortices at certain distance behind the body. Since the tightly rolled vortices are much stronger in strength than the distributed vortex sheets, the distributed vortex sheets can be neglected and the tightly rolled vortices can be simplified as a pair of concentrated vortices at the points A_1 and A_2. Although the fundamental mechanism for the generation of such vortices are of viscous origin, the basic flow features can be represented by an inviscid model with concentrated vortices once the vortices are established. The concentrated vortices behind the circular cylinder or a similar bluff body establish the recirculatory flow behind the body and coexist with the two separation points S_1 and S_2, as shown in the right part of Fig. 4.

In this paper, the bodies considered are symmetric with respect to the body axis parallel to the free stream flow, and the separation points S_1 and S_2 and the vortex points A_1 and A_2 are assumed to be symmetric with respect to this body axis. The flow velocity at

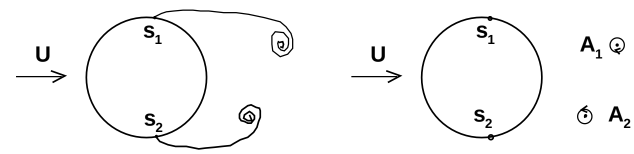

Fig. 4 Vortex Model for a circular cylinder. Left: continuous vortex sheets coming off from the cylinder; Right: inviscid model with two concentrated point vortices.

the postulated separation points S_1 and S_2 must be zero and must point towards them when approached from either side of the separation points.

As shown in the previous section, the location and strength of the stationary vortices are not unique in an inviscid model. For the circular cylinder, the vortices may lie anywhere on the Föppl line defined by Eqn. (6) with a corresponding vortex strength given by Eqn. (7). Each location of the vortices corresponds to a different separation point on the surface of the cylinder. The separation point can not be determined by the inviscid model. It has to be determined by considering viscosity effects of the fluid. However, provided that the separation point on the surface of the cylinder is given (e.g. based on experimental data), the location and strength of the stationary vortices can then be uniquely determined through Eqn. (19). In practical computations it is easier to select the vortex location and its strength through Eqn. (19) and then search for the location of the separation point on the body.

Fig. 5 shows the location of the stationary vortices and the vortex strength versus the location of the separation point for the circular cylinder case. r_0 in the figure is the distance of the vortices from the center of the cylinder. The separation angle θ_0 is measured anti-clockwise from the rear end of the cylinder. It is noticed from Fig. 5, that no stationary point of the vortices exists when the separation angle is equal to or greater than 90°. Fig. 6 replots D_0 and J_0 shown in Figs. 2 and 3 in terms of the separation angle θ_0 for the circular cylinder case under symmetric and anti-symmetric perturbations. The vortex system is less unstable as the separation points move farther upstream from the rear stagnation point.

It is known that when the boundary layer on the circular cylinder surface is laminar before separation, the separation point is on the windward side of the surface, i.e. θ_0 is greater than 90°, and when the boundary layer is turbulent before separation, the separation point is on the leeward side of the surface, i.e. θ_0 is smaller than 90° (see Roshko 1961[8]). Therefore, there exist no stationary symmetric separation vortices for the case of laminar separation. For the case

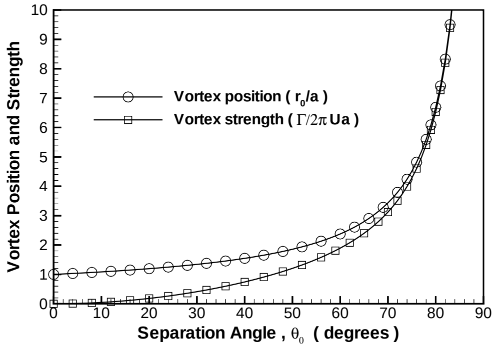

Fig. 5 Vortex position and strength vs. separation angle for circular cylinder.

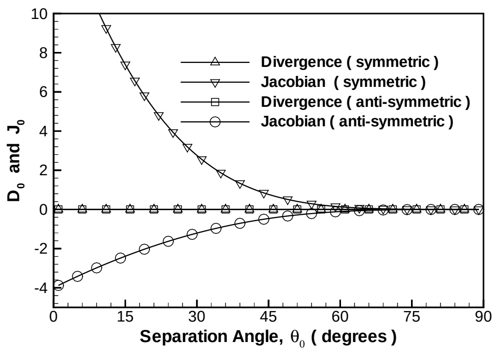

Fig. 6 Divergence and Jacobian vs. separation angle for circular cylinder under symmetric and anti-symmetric perturbations.

of turbulent separation, stationary symmetric separation vortices exist but are unstable under small anti-symmetric perturbations. This is consistent with the fact that only asymmetric vortices were observed behind a circular cylinder in the experiments by Roshko (1961).[8]

V. Vortices behind a Circular Cylinder with a Rear Splitter

In this section, the analysis is extended to include a splitter plate at the rear end of a circular cylinder of radius a, where the height of the splitter plate h is measured from the origin, i.e., the end point of the plate is at $x = h$ (see Fig. 7). In a three-dimensional situation, this corresponds to the case of having a vertical fin attached to a circular body. Since a splitter plate has no affect on the flow so long as the flow is symmetric, the result for the circular cylinder under symmetric perturbations remains unchanged for the case with a splitter plate, that is, the vortices will be neutrally stable. The splitter plate, however, will affect the flow under anti-symmetric perturbations. Intuitively, it reduces the 'communication' between the top and bottom vortices. It is interesting to see whether this reduction in 'communication' may reduce the instability due to anti-symmetric perturbations that exists in the circular cylinder case, and if so, whether there is a minimum length of the splitter plate that makes the vortices stable or neutrally stable.

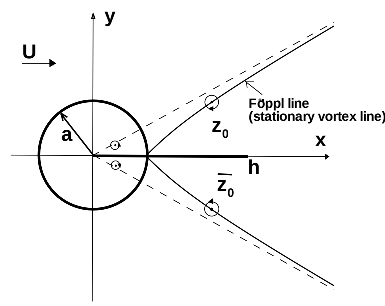

Fig. 7 Circular cylinder with a rear splitter plate.

The velocity field around a circular cylinder of radius a with the splitter plate in the complex plane $Z = x + iy$ can be easily found by using two consecutive conformal mappings from the flow past a circular cylinder of radius a_1 without the splitter plate in the plane $\zeta = \xi + i\eta$. These are

$$\zeta' = \frac{1}{2}(Z + \frac{a^2}{Z}) \tag{27}$$

and

$$\zeta' - X_m = \frac{1}{2}(\zeta + \frac{a_1^2}{\zeta}) \tag{28}$$

where

$$X_m = \frac{(a-h)^2}{4h} \tag{29}$$

and

$$a_1 = \frac{(a+h)^2}{4h} \tag{30}$$

The vortex strength does not change in the transformation.

Suppose the symmetric stationary vortices undergo small arbitrary displacements, i.e., Z_0 and $\overline{Z}_0$ change to Z_1 and Z_2, respectively. The circulation of the vortices Γ may be assumed unchanged for a short period of time right after the small displacement. When Z_1 and Z_2 are not symmetric to the real axis x, a vortex will be shed from the sharp edge of the splitter plate to satisfy the finite velocity condition at the sharp edge. Under small perturbation, however, the strength of this vortex is much weaker than that of the original symmetric vortex pair. Therefore, the shed vortex can be ignored in the following stability analysis for the anti-symmetric perturbation.

The vortex velocity at the vortex located at a point Z_1 is obtained by a limiting process (see Rossow 1978[9])

$$\begin{aligned} u - iv \quad = \quad & \Bigg[U\left(1 - \frac{a_1^2}{\zeta_1^2}\right) + \frac{i\Gamma}{2\pi}\left(-\frac{1}{\zeta_1 - a_1^2/\overline{\zeta}_1}\right. \\ & \left.\left. -\frac{1}{\zeta_1 - \zeta_2} + \frac{1}{\zeta_1 - a_1^2/\overline{\zeta}_2}\right)\right]\left(\frac{d\zeta}{dZ}\right)_1 \end{aligned}$$

$$
-\frac{i\Gamma}{4\pi}\left(\frac{d^2Z}{d\zeta^2}\right)_1\left(\frac{d\zeta}{dZ}\right)_1^2 \tag{31}
$$

where the subscript 1 denotes the values at $\zeta = \zeta_1$ or $Z = Z_1$,

$$
\frac{d\zeta}{dZ} = \frac{\zeta^2(Z^2-a^2)}{Z^2(\zeta^2-a_1^2)} \tag{32}
$$

and

$$
\frac{d^2Z}{d\zeta^2} = \frac{2}{(Z^2-a^2)\zeta^3}\left[a_1^2Z^2 - \frac{a^2Z^3(\zeta^2-a_1^2)^2}{(Z^2-a^2)^2\zeta}\right] \tag{33}
$$

For an anti-symmetric perturbation from the symmetric stationary positions Z_0 and $\overline{Z}_0$, we have

$$
\begin{cases} Z_1 = Z_0 + \Delta Z \\ \\ Z_2 = \overline{Z}_0 - \overline{\Delta Z} \end{cases} \tag{34}
$$

where $\Delta Z = \Delta x + i\Delta y$ is the perturbation, and $|\Delta x| \ll a$ and $|\Delta y| \ll a$. The transformed values or ζ_1 and ζ_2 can be obtained by Eqns. (27) and (28). The symmetry properties preserve under the conformal mapping as long as the perturbation is small.

The stability conditions in Section 3 is then applied to Eqn. (31) in terms of Z_1. Fourth order accurate finite-difference formulae are used in evaluating D_0 and J_0 with 64-bit arithmetics for extra high accuracy. Notice that the stationary points of the vortices for this case remain on the same Föppl line given by Eqn. (6) and shown in Fig. 1 for the circular cylinder case. We consider the particular case where the stationary point is at $x_0/a = \sqrt{55}/4$, $y_0/a = 3/4$, and $r_0/a = 2$. The corresponding vortex strength is $\frac{\Gamma}{Ua} = \frac{45}{16}\pi$ according to Eqn. (7). The corresponding separation point is at $\theta_0 = 53.4°$ according to Fig. 5.

As discussed at the end of Section 3, the divergence of the vortex velocity field for this case is zero. Direct numerical computations show values in the order of 10^{-20} for $U = 1$ and $a = 1$. Therefore, the stability of this case depends on the sign of J_0. The vortices are unstable when $J_0 < 0$, and neutrally stable when $J_0 \geq 0$. Fig. 8 shows the calculated divergence and Jacobian at the above x_0 and y_0 versus height of the splitter plate. Clearly, the divergence is zero for both symmetric and anti-symmetric perturbations. The symmetric result should be independent of the height of the splitter plate, which is confirmed by the constant positive J_0 shown in Fig. 8.

For the anti-symmetric perturbations, J_0 is less than zero for splitter plates that are shorter than approximately 1.7 times of the radius of the cylinder, i.e, $h/a < 2.7$. The vortices become neutrally stable when the length of the splitter plate increases beyond that critical value. As the length of the splitter plate goes to infinity, J_0 approaches the value of the symmetric perturbation because the splitter plate restraints the asymmetric motion of the vortices. It is also to

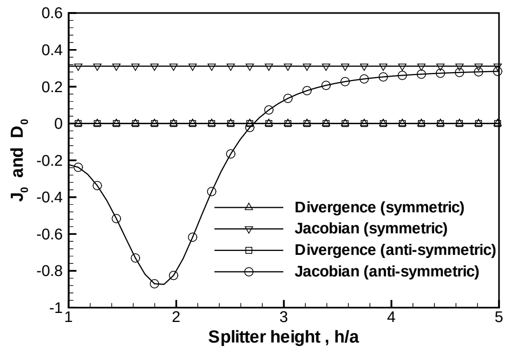

Fig. 8 Divergence and Jacobian vs. height of splitter plate for circular cylinder with a splitter plate, $r_0/a = 2$, $\theta_0 = 53.4°$.

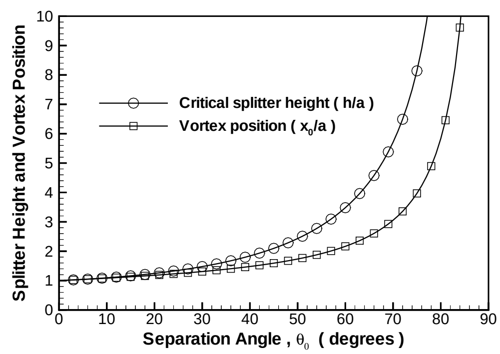

Fig. 9 Vortex position and critical height of splitter plate as function of separation angle for circular cylinder with a splitter plate.

be noticed that, as the length of the splitter plate increases from zero, the vortices initially become less stable (J_0 decreases) until the plate length reaches about 0.8 times the radius of the circular cylinder. At that point the value of J_0 begins to increase. It is not until the length of the plate reaches about $1.4a$ when the addition of the splitter plate enhances the stability compared to without the splitter plate. In other words, when the splitter plate is not high enough, it has an opposite effect, i.e., the addition of the splitter plate makes the symmetric stationary vortex pair more unstable under anti-symmetric perturbations. No two-dimensional experimental data have been found yet by the authors to support this finding. However, similar effects are predicted and shown to agree with known experimental observations for three-dimensional flows in Cai et al. (2001).[6]

Fig. 9 shows the vortex position x_0/a and the critical height h/a of the splitter plate as a function of the separation angle θ_0. Both h/a and x_0/a increase with the increase of θ_0, and the critical height h/a is always greater than x_0/a. Physically, the splitter plate has to extend further downstream than the vortices in order to cut-off the 'communication' between the two vortices as the vortices move away from the cylinder.

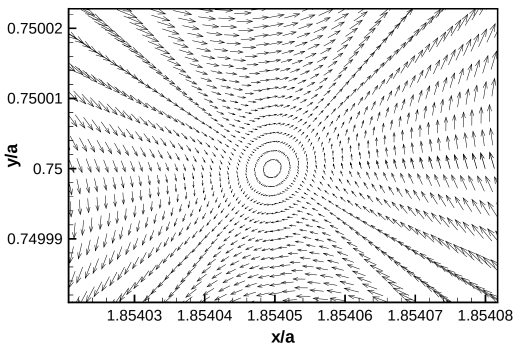

Fig. 10 Velocity of vortex under anti-symmetric perturbations for the flow over a circular cylinder with a splitter plate, $h/a = 2$, $r_0/a = 2$ (unstable case).

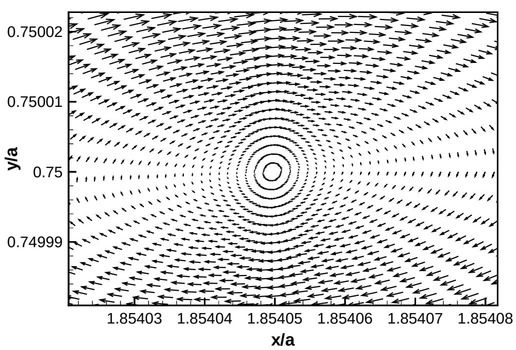

Fig. 11 Velocity of vortex under anti-symmetric perturbations for the flow over a circular cylinder with a splitter plate, $h/a = 4$, $r_0/a = 2$ (neutrally stable case).

This agrees with the experimental results of Roshko (1961),[8] which showed that the main effect of the splitter plate with $h/a = 6.3$ and θ_0 being about 75° is the suppression of the alternating vortex shedding and removal of the peak frequency in the spectrum. Fig. 9 predicts that the critical height of the splitter plate is $h/a = 8.1412$ for $\theta_0 = 75°$. The above results provide guidance in the choice of the height of the splitter plate to suppress the alternating vortex shedding behind the circular cylinder.

Although the formula to calculate the vortex velocity for this case are different and more complex than those for the pure circular cylinder case, it is expected that the results for this case must reduce to those of the pure circular cylinder case as the length of the splitter plate becomes zero. Indeed, from Fig. 8, the calculated value of J_0 is 0.311523 for symmetric perturbations and -0.223630 for anti-symmetric perturbations when $h/a = 1$, which are identical to the analytical values for the circular cylinder to the 6-th significant digit.

In order to appreciate the physical meaning of the stability conditions in terms of D_0 and J_0, Figs. 10 and 11 show the vortex velocity fields for anti-symmetric perturbations at $r_0/a = 2$ for the plate heights $h/a = 2$ and $h/a = 4$, respectively. From Fig. 8, it is known that the former is unstable and the latter is neutrally stable. Clearly, Figs. 10 and 11 show a zero vortex velocity at the stationary vortex point under consideration. Since the divergence of the velocity field $D_0 = 0$, some velocities must move out this stationary point if others move in, which is the case when $J_0 < 0$. It is the existence of the diverging velocities that lead to instability.

For the case $h/a = 4$, the splitter plate is long enough to make the vortex neutrally stable under anti-symmetric perturbations. Since D_0 is still zero, $J_0 > 0$ indicates that the velocity vectors form circles around the stationary point as is confirmed by Fig. 11. In this case, the vortex is in an oscillatory, neutrally stable condition around the stationary point.

As pointed out earlier, it is conjectured that the divergence of the velocity field of the motion of a vortex center in a two-dimensional incompressible flow is always zero. Therefore, vortices in a two-dimensional incompressible flow can at best be neutrally stable. However, if D_0 might be less then zero, there would exist the possibility that the vortex velocity vectors may all be drawn towards the stationary point when $J_0 > 0$, leading to a stable vortex configuration. Such situations exit in three-dimensions discussed in Cai et al. (2001).[6] On the other hand, when $D_0 > 0$, the vortex must be unstable regardless of the value of J_0 because of the fact that some velocity vectors must leave the stationary point when the divergence is positive.

VI. Vortices Behind an Elliptic Cylinder

Consider the flow shown in Fig. 12 where the major axis of the ellipse is perpendicular to the flow direction. The lengths of the major and minor axes of the ellipse are $2b$ and $2c$, respectively. The thickness ratio of the elliptic cylinder is difined as $\tau = c/b$. The two vortices behind the elliptic cylinder are at the $Z_0 = x_0 + iy_0$ and $\overline{Z}_0 = x_0 - iy_0$.

The vortex velocity field for this case is given by the same formula as Eqn. (31) with U replaced by $U/2$ and $a_1 = 1$. However, the transformation from ζ to Z is different for this case.

$$Z = \frac{1}{2}\left(\zeta + \frac{\lambda}{\zeta}\right) \tag{35}$$

where $c = \frac{1+\lambda}{2}$ and $b = \frac{1-\lambda}{2}$. Thus

$$\frac{d\zeta}{dZ} = \frac{2\zeta^2}{\zeta^2 - \lambda} \tag{36}$$

$$\frac{d^2 Z}{d\zeta^2} = \frac{\lambda}{\zeta^3} \tag{37}$$

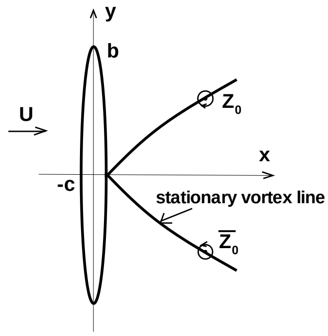

Fig. 12 Stationary vortex line behind an elliptic cylinder, $\tau = 0.1$.

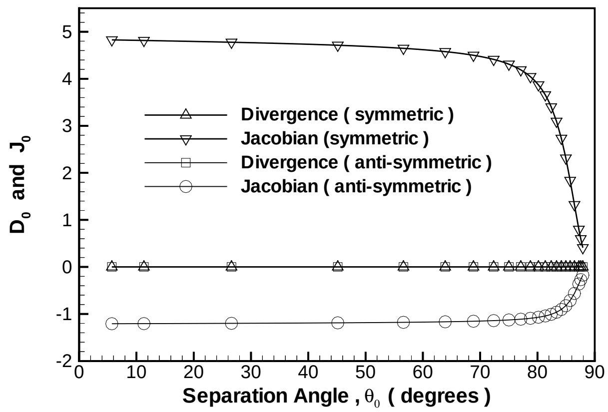

Fig. 14 Divergence and Jacobian vs. separation angle for an elliptic cylinder, $\tau = 0.1$.

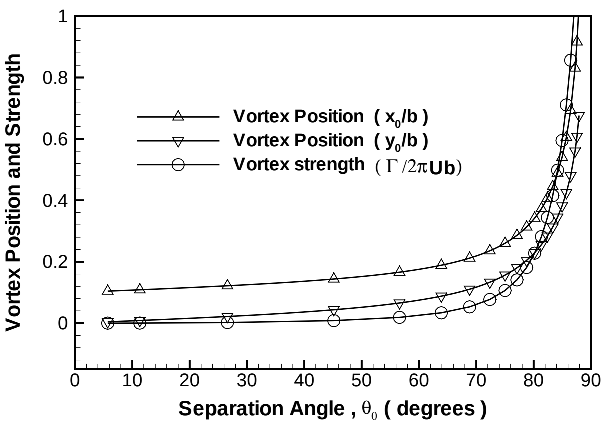

Fig. 13 Vortex position and strength vs. separation angle for an elliptic cylinder, $\tau = 0.1$.

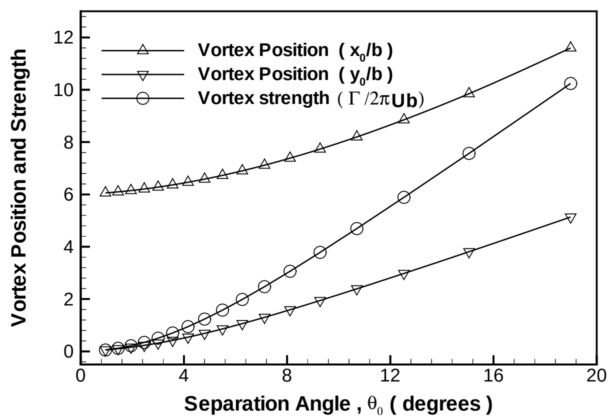

Fig. 15 Vortex position and strength vs. separation angle for an elliptic cylinder, $\tau = 6.0$.

Fig. 13 shows the relation between the flow separation angle on the surface of the elliptic cylinder and the stationary position and the strength of the vortices for the ellipse with $\tau = 0.1$. Again, the separation angle is measured anti-clockwise from the rear stagnation point of the cylinder. The lines of stationary vortex points for this case are shown in Fig. 12. As in the circular cylinder case, when a flow has separation point close to 90°, the stationary vortices must be placed far away from the cylinder. This means that it is not possible to find stationary points of the vortices when separation occurs at the top and bottom extremes of the elliptic cylinder. This agrees with the finding by Smith and Clark (1975)[10] that no stationary points exist for a flat plate normal to the flow where separation must occur at the ends of the plate, i.e., $\theta_0 = 90°$, in order to satisfy the Kutta condition. The ellipse approaches a vertical flat plate as $\tau \to 0$.

Fig. 14 shows the D_0 and J_0 for this case. It is noticed again that D_0 is zero for both the symmetric and anti-symmetric perturbations. Therefore, the vortex system is at most neutrally stable. For anti-symmetric perturbations, J_0 is always less than zero. Therefore, the vortices are unstable to small anti-symmetric perturbations. On the other hand, J_0 is always greater than zero for symmetric perturbations, and thus the vortices are neutrally stable to small symmetric perturbations. These conclusions about an elliptic cylinder are qualitatively identical to those for the circular cylinder case.

Figs. 15 and 16 show the results for an elliptic cylinder with $\tau = 6$. It is seen that the stationary symmetric vortex pair is unstable under small anti-symmetric perturbations, and thus may lead to possible asymmetric configurations or asymmetric vortex shedding. This agrees with the wind-tunnel test results (Bradshaw 1970, also Van Dyke 1982, Fig. 32, p. 24)[11] at a Reynolds number 4000 based on $2c$. The experimental results show that a laminar boundary layer separates at $\theta_0 \approx 8°$ and sheds alternating vortices.

VII. Conclusions

Föppl's classical problem of the stability of a pair of vortices behind a circular cylinder is revisited. An error in his analysis and conclusion on the stabil-

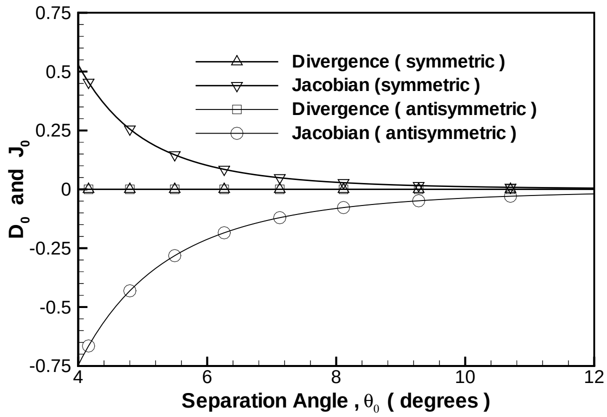

Fig. 16 Divergence and Jacobian vs. separation angle for an elliptic cylinder, $\tau = 6.0$.

ity of vortices under small symmetric perturbations is pointed out. A general stability condition for vortices in an arbitrary two-dimensional flow is derived. This condition can be easily applied analytically or numerically for complex flow problems. It is applied to the problem of symmetric vortices behind circular cylinders with or without rear splitter plates and also elliptic cylinders. The following conclusions are drawn from the current studies.

1. The stability of any vortex or a vortex system for any given mode of small motion in a two-dimensional space can be determined by calculating the Jacobian and divergence of the vortex velocity field in terms of the location of the vortex under consideration. The stability conditions are listed in Table 1.

2. It is conjectured that the divergence of the vortex velocity field in a two-dimensional incompressible flow is always zero. Thus, vortices in two-dimensional incompressible flows are at best neutrally stable according to the new stability conditions.

3. In an inviscid model with two concentrated symmetric vortices for the two-dimensional flow over a circular or elliptic cylinder, no stationary points of vortices exist for which the separation angle is equal or greater than 90 degrees measured from the rear stagnation point of the cylinder.

4. A pair of stationary symmetric vortices behind a circular or elliptic cylinder is neutrally stable under small symmetric perturbations, and it is unstable under small anti-symmetric perturbations.

5. Adding a splitter plate of sufficient length behind a circular cylinder makes the vortices neutrally stable under small anti-symmetric perturbations. However, a splitter plate of a short length has opposite effects.

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